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ASYMPTOTIC BEHAVIOUR OF THE SCATTERING AMPLITUDE AT HIGH ENERGIES

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Abstract: The asymptotic behaviour of the scattering amplitude is investigated at high energies. It is shown that at high energies the usual diffraction picture of scattering contradicts the unitarity conditions and analytic properties of the scattering amplitude formulated with the help of Mandelstam's representations. In terms of these conditions it is natural to expect that the cross section should decrease faster than $1/\ln E$.

1. Introduction

The asymptotic behaviour of scattering amplitudes in quantum field theory has been investigated in refs. [1-3]. In all cases, however, only rather general restrictions on possible asymptotics have been obtained. At present, mainly owing to Mandelstam [4], it has become possible in studying asymptotic behaviours to make a more extensive use of unitary conditions and the dispersion relations for the momentum transfer. Since thus far we have been able to deal only with two-body states under unitarity conditions, we can hardly expect a complete solution of the problem. Nevertheless, some limited information can be obtained, as will be shown below.

The description of elastic scattering at high energies is based on the so-called diffraction picture. According to this picture, particles with an impact parameter ρ smaller than a certain R (of the order of $1/\mu$, μ being the meson mass) interact strongly with the scatterer and are discharged from the elastic channel, while particles with an essentially larger impact parameter are not scattered. This leads to a diffractional scattering with two main characteristics: both the total scattering cross-section σ_n and the differential cross-section for elastic scattering per unit interval of the square of the momentum transfer $d\sigma/dt$ are energy independent ($-t$ being the square of the momentum transfer).

The present paper investigates the consistency of this picture with the requirements of unitarity and analyticity for scattering amplitudes. It is shown that energy independence of σ_n and $d\sigma/dt$ cannot be reconciled with the conditions of unitarity and analyticity. In terms of these conditions it is natural to expect that σ_n and $d\sigma/dt$ decrease faster than logarithmically.

Investigating inelastic processes in the pole approximation, V. Berestetsky and I. Pomeranchuk [5]) have arrived at a similar inconsistency of diffractional representations with the general properties of amplitudes in quantum field theory. Assuming that the elastic scattering cross-section remains constant, they concluded that the cross-section of inelastic processes tends to infinity. An assumption of a decrease of the scattering cross-section makes their results self-consistent.

2. General Properties of the Scattering Amplitude

For the sake of simplicity we shall first consider the scattering of two identical particles without spin (mesons) and shall assume that these particles are the lightest in the theory and are pseudoscalar.

The elastic scattering amplitude of these particles will be treated as a function of the two invariant variables: the square of the centre-of-mass energy S and the square of the momentum transfer $-t$. The quantity $A = A(S, t)$ is normalised in such a way that

$$A_1(S, 0) \equiv \operatorname{Im} A(S, 0) = \frac{S}{16\pi}\sigma_n \qquad (1)$$

we shall assume that $A(S, t)$ possesses the analytic properties formulated by Mandelstam [4]) and shall use the Mandelstam plane (fig. 1).

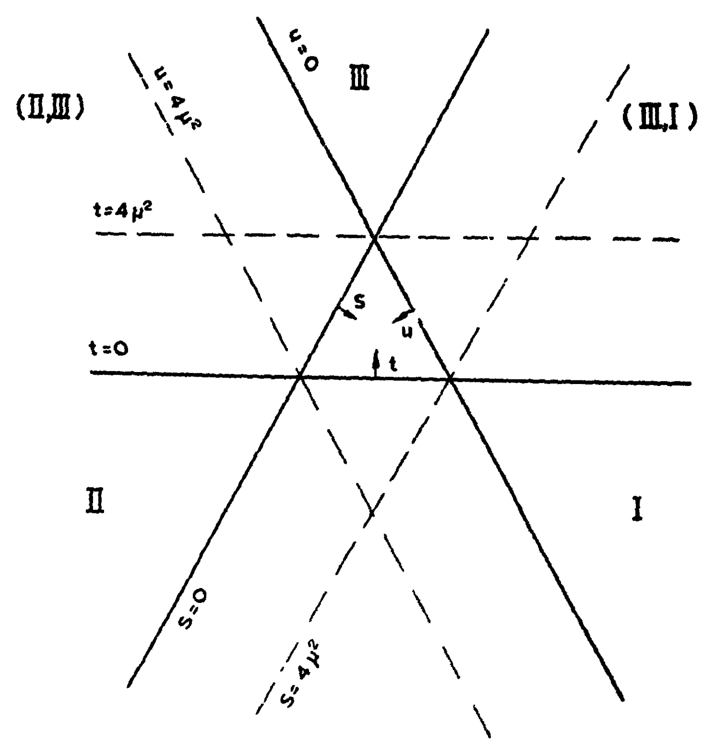

Fig. 1. The Mandelstam plane.

The functions $A(S, t)$ and $A_1(S, t)$ will interest us in the region I at $S \to \infty$, $-t \ll S$. In the $t < 4\mu^2$ region $A(S, t)$ and $A_1(S, t)$ can be represented as a series of Legendre polynomials. In particular, $A_1(S, t)$ can be written down in the form

$$A_1(S, t) = \sqrt{\frac{S}{S-4\mu^2}} \sum_{l=0}^{\infty} a_l(S)(2l+1)P_l\left(1+\frac{t}{2p^2}\right), \tag{2}$$

where $0 < a_l < 2$; $p = \frac{1}{2}\sqrt{S-4\mu^2}$ is the particle momentum in the centre-of-mass system.

If we use the dispersion relation for $A_1(S, t)$ as a function of "t"

$$A_1(S, t) = \frac{1}{\pi}\int_{t_1(S)}^{\infty} \frac{\operatorname{Im} A_1(S, t')\mathrm{d}t'}{t'-t} - \frac{1}{\pi}\int_{-\infty}^{t_2(S)} \frac{\operatorname{Im} A_1(S, t')\mathrm{d}t'}{t'-t}, \tag{3}$$

where

$$t_1(S) > 4\mu^2, \quad t_2(S) < -S,$$

then according to ref. [6]) we have

$$a_l(S) = \frac{4}{\pi\sqrt{S(S-4\mu^2)}} \int_{t_1(S)}^{\infty} Q_l\left(1+\frac{t'}{2p^2}\right) \operatorname{Im} A_1(S, t')\mathrm{d}t', \tag{4}$$

with

$$Q_l(z) = \frac{1}{2}\int \frac{P_l(z')}{z'-z}\mathrm{d}z'.$$

From this formula one can easily derive [6]) the asymptotic behaviour of $a_l(S)$ at $l \to \infty$. Since at $l \gg 1$ and $t'/2p^2 \ll 1$ we have

$$Q_l\left(1+\frac{t'}{2p^2}\right) = K_0(\rho\sqrt{t'}),$$

where $\rho = l/p$ is the impact parameter and $K_0(x)$ is the McDonald function, the quantity

$$a_l(S) \equiv a(\rho, S) = \frac{4}{\pi}\int_{4\mu^2}^{\infty} K_0(\rho\sqrt{t'}) \operatorname{Im} A_1(S, t')\frac{1}{S}\mathrm{d}t', \quad \frac{S}{4\mu^2} \gg 1 \tag{5}$$

decreases exponentially at $\rho \gg 1/2\mu$ and any S, provided only that $\operatorname{Im} A_1(S, t')$ does not essentially change at $t-4\mu^2 \ll 4\mu^2$.

At this point the analytic properties of $A_1(S, t')$ are consistent with the diffraction picture. Exponential decrease of $a_l(S)$ at $l \gg p/2\mu$ means that the function $A_1(S, t)$ changes essentially only with a change of t by a value greater than or of the order of μ^2. This is a simple consequence of the uncertainty relation between l and scattering angle ϑ.

At $S \to \infty$ there are three possibilities:

$$a(\rho, S) \to a(\rho), \tag{6a}$$

$$a(\rho, S) \to 0, \tag{6b}$$

$a(\rho, S)$ oscillates around the mean value of $a(\rho)$ (6c). Passing from summation to integration in (2) and employing the relation (at $l \gg 1$ and $t/2p^2 \ll 1$)

$$P_l\left(1+\frac{t}{2p^2}\right) = J_0\left(\frac{l}{p}\sqrt{-t}\right) \equiv J_0(\rho\sqrt{-t}), \tag{7}$$

where $J_0(x)$ is the Bessel function, we shall obtain simple relations for $A_1(S, t)$ at $S \to \infty$ in each of these cases.

$$A_1(S, t) = \begin{cases} \frac{1}{4}S\int a(\rho, S)2\rho\, \mathrm{d}\rho J_0(\rho\sqrt{-t}) & \text{at} \quad t < 0, \\ \frac{1}{4}S\int a(\rho, S)2\rho\, \mathrm{d}\rho I_0(\rho\sqrt{t}) & \text{at} \quad 0 < t < 4\mu^2, \end{cases} \tag{8}$$

where $I_0(x)$ is the Bessel function of imaginary argument.

In the first case

$$A_1(S, t) = Sf(t), \quad \sigma_n = 16\pi f(0) = \text{const.} \tag{9}$$

In the second case

$$A_1(S)/S \to 0, \quad \sigma_n \to 0. \tag{10}$$

In the third case

$$A_1(S, t) = Sf(S, t), \tag{11}$$

where $f(S, t)$, although it oscillates with S, as a function of t changes essentially only with changes in t of the order of μ^2.

Only the first case is completely consistent with the usual diffraction picture. Note that, if in this case we assume (6a) to hold at any S, then, by virtue of (5), $\mathrm{Im}\, A_1(S, t) = Sf(t)$, at least for $t - 4\mu^2 \ll \mu^2$ (The author's attention to the fact that $A_1 = Sf(t)$ corresponds to the diffraction picture and that this expression must be fitted with the dispersion relation on t was drawn by I. Pomeranchuk.)

So far we have concerned ourselves only with analytic properties of $A_1(S, t)$ as a function of t and employed the unitarity condition in the physical region on a limited scale (the $0 < a_l < 2$ condition). However, as Mandelstam [4]) first showed, unitarity conditions in other physical regions also impose essential restrictions on amplitudes by virtue of the analytic property of $A(S, t)$ as a function of S.

Let us consider the unitarity condition in region (III) (fig. 1). At $4\mu^2 < t < 16\mu^2$, inelastic processes in channel t being forbidden, the unitarity condition has a simple form:

$$A_3(S,t) = \frac{1}{4\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\frac{dz_1\,dz_2}{\sqrt{1+2zz_1z_2-z_1^2-z_2^2-z^2}}A(S_1,t)A^*(S_2,t), \quad (12)$$

where $A_3(S,t) = \operatorname{Im} A(S,t)$ in region (III); $z = 1+2S/(t-4\mu^2)$ is the cosine of scattering angle in the centre-of-mass system of channel (III):

$$z_{1,2} = 1+2S_{1,2}/(t-4\mu^2).$$

This form of the unitarity condition differs from the usual one only by replacement of the variable φ_1 by z_2. Integration is performed over the region in which the radical is positive. It was shown by Mandelstam that this relation can be continued from region III ($z < 1$) to regions I, III ($z > 1$). In continuing (12) sufficiently far to regions I, III, expressions to the right and to the left become complex. Calculating their imaginary part and taking into account that $\operatorname{Im} A_1(S,t) = \operatorname{Im} A_3(S,t) = \rho(S,t)$ we shall obtain, according to Mandelstam,

$$\operatorname{Im} A_1(S,t) = \frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\frac{dz_1 dz_2}{\sqrt{z^2-2zz_1z_2+z_1^2+z_2^2-1}}A_1(S_1,t)A_1^*(S_2,t) \quad (13)$$

at $4\mu^2 < t < 16\mu^2$. Integration is performed over the domain

$$z_1z_2+\sqrt{(z_1^2-1)(z_2^2-1)} < z. \quad (14)$$

Note that this equation is valid irrespective of the asymptotic behaviour of $A_1(S,t)$ at $S \to \infty$.

3. Investigation of Behaviour of $A_1(S,t)$ for large S using the Unitarity Condition

Since $A_1 = Sf(t)$ is the most natural expression for $A_1(S,t)$ obtained on the basis of diffraction picture, it is necessary first of all to substitute it into the unitarity condition and to check wheter it can be fulfilled.

This operation is simple enough since for such an asymptotic behaviour of A_1 and for $z \gg 1$, an essential contribution to the integrand is made by $z_1 \gg 1$, $z_2 \gg 1$. Therefore it is possible to substitute $A_1(S_1,t) = S_1f(t)$ and $A_1^*(S_2,t) = S_2f^*(t)$ into the right-hand side of eq. (13). Then integrating we obtain

$$\operatorname{Im} A_1(S,t) = \frac{1}{6\pi}\sqrt{\frac{t-4\mu^2}{t}}(t-4\mu^2)S\ln S|f(t)|^2+SO(1). \quad (15)$$

Since the left-hand side must equal $S \operatorname{Im} f(t)$ under the assumption, then in view of the presence of $\ln S$ in the right-hand side we come to the conclusion that $A_1(S, t) \neq Sf(t)$ for arbitrarily large S.

Since $A_1 = Sf(t)$ is impossible, it is necessary to find out what the character of a function satisfying eq. (13) can generally be. For this purpose we shall apply the Mellin transformation † to eq. (13), i.e. multiply the right and the left-hand sides by $z^{-(p+1)}$ and integrate over z. $A(S, t)$ does not have an essential singularity at $S \to \infty$, therefore $A_1(S, t)$ can be written in the form

$$A_1(S, t) = S^{q(t)} B_t(S), \tag{16}$$

where $B_t(S)$ neither increases nor decreases as a power at $S \to \infty$. Thanks to this, the Mellin transformation makes sense at any $p > q$. At $z = 0$ there is no divergence as $A_1(S, t) = 0$ at $z < z_0$, $z_0 > 1$. Integrating over z we obtain

$$\begin{aligned}&\int \mathrm{Im}\, A_1(S, t) z^{-(p+1)} \mathrm{d}z \\ &= \frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}} \int \mathrm{d}z_1 (z_1^2-1)^{-(p+1)/2} A_1(S_1, t) \int \mathrm{d}z_2 (z_2^2-1)^{-(p+1)/2} A_1^*(S_2, t) c(\alpha),\end{aligned} \tag{17}$$

where

$$\alpha = \frac{z_1 z_2}{\sqrt{(z_1^2-1)(z_2^2-1)}} \quad \text{and} \quad c(\alpha) = \int_1^\infty \frac{(x+\alpha)^{-(p+1)}}{\sqrt{x^2-1}} \mathrm{d}x.$$

Let us now consider this equation for p in the neighbourhood of q, $p = q + \delta$. In this case if $B_t(S)$ does not decrease at $S \to \infty$, then in the right hand side of (17) $z_1 \gg 1$ and $z_2 \gg 1$ become essential for $\delta \ll 1$. Under these conditions $\alpha \to 1$. Denoting

$$\varphi(p) = \int z^{-(p+1)} A_1(S, t) \mathrm{d}z, \tag{18}$$

we then obtain

$$\mathrm{Im}\, \varphi(p) = \frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}}\, c(1) |\varphi(p)|^2. \tag{19}$$

But since $|\mathrm{Im}\, \varphi(p)| < |\varphi(p)|$ it follows from (19) that

$$|\varphi(p)| < \frac{\pi}{c(1)}\sqrt{\frac{t}{t-4\mu^2}} \tag{20}$$

at $\delta \ll 1$. This means that

$$\int z^{-\delta} z^{-q} A_1(S, t) \frac{\mathrm{d}z}{z} < \infty$$

for any $\delta \ll 1$, i.e.

$$\int^\infty B_t(S) \frac{\mathrm{d}S}{S} < \infty. \tag{21}$$

† The possibility of investigating eq. (13) with the help of the Mellin transformation was pointed out to the author by L.D. Landau.

The integral (21) can converge either due to the decrease of $B_t(S)$ or due to oscillations. It can be easily proved by direct substitution that a purely oscillatory solution does not satisfy eq. (13). Therefore we shall interpret the condition (21) as the condition that $B_t(S)$ decrease faster than $1/\ln S$. Since, apart from this, $B_t(S)$ by definition is not power dependent at $S \to \infty$, it is more convenient to regard $B_t(S)$ as a function of $\xi = \ln S$ and instead of (16) to write down

$$A_1(S, t) = S^q B_t(\xi), \quad \int B_t(\xi)\mathrm{d}\xi < \infty. \tag{16a}$$

Thus we find that in order to make eq. (13) valid it is necessary for $A_1(S, t)$ to have the form of (16a) at $4\mu^2 < t < 16\mu^2$.

In this section we have so far been concerned with the behaviour of $A_1(S, t)$ as a function of S. Now let us see what can be deduced from (16a) on the behaviour of $A_1(S, t)$ as a function of t. If the order of the power in (16a) is a function of t, then $A_1(S, t)$ will change essentially with the change in t not by the order μ^2 but by the order $\mu^2/\ln S$, i.e., arbitrarily fast for sufficiently great S. Such a behaviour of $A_1(S, t)$ can hardly be considered possible, because firstly, for this behaviour of $A_1(S, t)$ the amplitudes $a_l(S)$ calculated by (5) would not, generally speaking, decrease exponentially for $l \gg p/2\mu$, and secondly, it seems extremely unlikely that a function, which in the region $t < 4\mu^2$ changes essentially only with a change of t of the order of μ^2 would begin to change arbitrarily fast when analytically continued beyond the branch point into the region $t > 4\mu^2$. In any case it seems reasonable to postopone the treatment of such rapidly changing functions until a more detailed investigation is carried out and to assume that q is independent of t and is equal to unity. Assuming that $B_t(\xi)$ also changes essentially only with a change of t of the order μ^2 we shall come to the conclusion that the third case (6), (11) is impossible too and that

$$A_1(S, t)/S \to 0 \quad \text{at} \quad S \to \infty. \tag{22}$$

So far we have shown only that the decrease of $B_t(\xi)$ is a necessary condition of $A_1(S, t)$ satisfying eq. (13). It is easy to show that there exists a solution of such a structure. We shall assume that $B_t(\xi)$ is a power-type function of ξ, i.e. it possesses the property that $B_t(\xi_1+\xi_2) = B_t(\xi_1)$ if $\xi_1 \gg \xi_2$, and we shall put for simplicity $q = 1$ (the same result is easily arrived at in case of arbitrary q).

Contrary to (15), in the case of the asymptotic behaviour of (16a) regions $z_1 \gg 1$, $z_2 \sim 1$; $z_1 \sim 1$, $z_2 \gg 1$ prove to be essential for the integrand in the right-hand side of (13) at $z \gg 1$. Therefore an asymptotic expression for $A_1(S, t)$ cannot be directly substituted into it.

We shall write down (13) in the following way

$$\operatorname{Im} A_1(S, t) = \frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}} \int_{z_0^2}^{u_m(z)} \mathrm{d}u \int_{z_0}^{u/z_0} \frac{\mathrm{d}z_1}{z_1} \frac{A_1(z_1)A_1{}^*(u/z_1)}{\sqrt{z^2-2zu+u^2/z_1{}^2+z_1{}^2-1}}, \tag{23}$$

where

$$u = z_1 z_2, \quad A_1(z) \equiv A_1(S, t) \quad \text{with} \quad A_1 = 0 \quad \text{at} \quad z < z_0.$$

The main contribution to the integral over u is made by $u \approx z$. Therefore, the integral over z_1 can be broken into two parts from z_0 to λ and from λ to u/z_0 and λ can be chosen in such a way that

$$z_0 \ll \lambda \ll u/z_0.$$

In the integral from z_0 to λ, $u/z_1 \gg 1$ and therefore we may substitute for $A_1^*(u/z_1)$ the asymptotic expression

$$\frac{u}{z_1} B_t^* \left(\ln \frac{u}{z_1}\right) \tfrac{1}{2}(t-4\mu^2) \approx \tfrac{1}{2}(t-4\mu^2) \frac{u}{z_1} B_t(\ln u).$$

Then we obtain

$$\tfrac{1}{2}(t-4\mu^2)u \int_{z_0}^{\lambda} \frac{dz_1}{z_1} \frac{A_1(z_1)}{z_1\sqrt{z^2-2zu+u^2/z_1^2+z_1^2-1}} B_t^*(\ln u).$$

Since $\int A_1(z) dz_1/z_1^2$ converges according to the assumption, the main contribution to the integral from z_0 to λ is made by $z_1 \approx 1$ and therefore one can neglect $z_1^2 - 1$ as compared with the remaining terms and integrate not up to λ but up to the zero of the radical.

In the same way we may treat the integral from λ to u/z_0. It will yield the complex conjugate expression. Transforming to the original variables, (22) can be written in the form

$$\operatorname{Im} A_1(S, t) = \frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}} \int \frac{dz_1\, dz_2}{\sqrt{z^2-2zz_1z_2+z_2^2}} A_1(S_1, t) B_t^*(\ln S_2) S_2$$

$$+\frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}} \int \frac{dz_1\, dz_2}{\sqrt{z^2-2zz_1z_2+z_1^2}} S_1 B_t(\ln S_1) A_1^*(S_2, t), \quad (24)$$

where integration is performed respectively over the regions:

$$z_2(z_1+\sqrt{z_1^2-1}) \leqq z, \quad z_1(z_2+\sqrt{z_2^2-1}) \leqq z.$$

After this transformation the first integral is easily integrated over z_2 and the second-over z (certainly asymptotically at $S \to \infty$).

As a result we obtain

$$\operatorname{Im} A_1(S, t) = SB_t(\xi)\varphi^*(t) + SB_t^*(\xi)\varphi(t), \quad (25)$$

where

$$\varphi(t) = \frac{1}{2\pi}\sqrt{\frac{t-4\mu^2}{t}} \int_{z_0}^{\infty} dz' A_1(S', t) \left[z' \ln \frac{z'+1}{z'-1} - 2\right]$$

$$= \frac{1}{2\pi}\sqrt{\frac{t-4\mu^2}{t}} \int_{-1}^{1} z dz \int_{z_0}^{\infty} \frac{A_1(S, t)}{z'-z} dz'. \quad (26)$$

The integral (26) converges provided eq. (16a) is fulfilled. Eq. (25) shows that (13) is really satisfied if $A_1(S, t)$ has the form of (16a). Simultaneously an equation for $B_t(\xi)$ emerges as a function of t. This equation holds only at $4\mu^2 < t < 16\mu^2$ but if we assume it to be valid for any t we obtain immediately

$$B_t(\xi) = B(\xi) \exp\left[(t-4\mu^2)\int_{4\mu^2}^{\infty} \frac{\delta(t')}{t'-t}\,\frac{dt'}{t'-4\mu^2}\right], \tag{27}$$

where $\exp[2i\delta(t)] = (1-2i\varphi(t))(1+2i\varphi^*(t))^{-1}$, and it is assumed that $\delta(t)$ does not decrease at $t \to \infty$. Thus, in the physical region $A_1(S, t) = SB(\xi)f(t)$ we have

$$\sigma_n = B(\xi)16\pi f(0) < \frac{c}{\xi} \quad \text{at} \quad S \to \infty, \tag{28}$$

while $d\sigma/\sigma_n\, dt$ is independent of S.

If from the outset we had proceeded from the assumption that $A_1(S, t) = \phi(S)f(t)$, then without making any further assumptions we would have arrived at the fact that $\phi(S) = SB(\xi)$ and that $f(t)$ satisfies the equation

$$\operatorname{Im} f(t) = f(t)\varphi^*(t) + f^*(t)\varphi(t). \tag{29}$$

It is noteworthy that the value $\varphi(t)$ (26) is in a certain way related to the scattering phase shifts at centre-of-mass system energy $\sqrt{t}$. Indeed, the scattering amplitude in region (III) can be written as

$$A(z, t) = \frac{1}{\pi}\int_{z_0}^{\infty} \frac{A_1(z', t)}{z'-z}\, dz' - \frac{1}{\pi}\int_{-\infty}^{-z_0} \frac{\operatorname{Im} A(z', t)}{z'-z}\, dz', \tag{30}$$

where z is the cosine of the scattering angle in the centre-of-mass system. If we now wanted to evaluate the p-wave amplitude by multiplying $A(z, t)$ by z and integrating over z we would obtain zero by virtue of the symmetry of $A(,z\, t)$. However, if we take only the first term in (30), multiply it by z and carry out the integration, we obtain $\varphi(t)$ with an accuracy up to the factor.

4. On the Energy Value at which the Cross-Section Decrease may prove to be essential

It is very difficult to make a definite estimate of energies at which the cross-section decrease may prove to be essential since we can use the unitary condition only in the interval $4\mu^2 < t < 16\mu^2$, which contains no contribution from the inelastic processes in the third channel.

However, one may attempt to make a rough estimate of the critical energy. It is rather interesting that such an estimate involves a numerical parameter which renders this energy very great.

In order to make this estimate we shall assume that starting with $S = A$ the cross-section becomes constant and $A_1 = Sf(t)$. Substituting $A_1 = Sf(t)$ into the unitarity condition (13) we arrive at eq. (15). To estimate up to what energy $A_1 \approx Sf(t)$, it is necessary to determine up to what energy the first term in (15) containing $\ln S$ remains small as compared with the entire $\operatorname{Im} A_1$. We shall see then that the cross-section may remain constant up to energies at which

$$\frac{1}{6\pi}\sqrt{\frac{t-4\mu^2}{t}}\,(t-4\mu^2)|f|^2 \ln\frac{S_0}{\Lambda} \ll \operatorname{Im} f(t). \tag{31}$$

If we now put $\operatorname{Im} f \approx f$, neglect the fact that $f(t)$ increases with t in the interval from 0 to $4\mu^2$ and substitute $\sigma_n/16\pi$ for $f(t)$, then instead of (31) we obtain

$$\frac{1}{96}\sqrt{\frac{t-4\mu^2}{t}}\,\frac{(t-4\mu^2)\sigma_n}{\pi^2}\ln\frac{S_0}{\Lambda} \ll 1. \tag{32}$$

Eq. (32) makes sense up to $t = 16\mu^2$, and therefore if $\sigma_n \approx 1/\mu^2$, eq. (32) means roughly that

$$\ln (S_0/\Lambda) \ll 100. \tag{33}$$

In spite of the fact that the value S_0 proves to be very large and that the cross-section possibly tends to zero only at unattainably high energies, the first term in (15) may result in a 10 % change of the cross-section already with a change of energy from 10^9 eV to 10^{13} eV.

5. Pion-Pion Scattering at High Energies

So far we have considered scattering of neutral particles without spins. In this section we shall show that the same reasoning can easily be applied to the case of pion-pion scattering.

In a following paper it will be shown that the same holds for pion-nucleon and nucleon-nucleon scattering. The scattering amplitude describing scattering of π-mesons with momentum p_1 in isobaric state α by π-mesons with momentum p_2 and isobaric state β into π-mesons with momenta $-p_3$, $-p_4$ and isobaric states γ, δ can be written in the form

$$T_{\alpha\beta,\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta}A(S_{12}, S_{13}, S_{23}) + \delta_{\alpha\gamma}\delta_{\beta\delta}A(S_{13}, S_{12}, S_{23}) + \delta_{\alpha\delta}\delta_{\beta\gamma}A(S_{23}, S_{12}, S_{13}). \tag{34}$$

In virtue of the crossing symmetry we have

$$A(x; y, z) = A(x; z, y). \tag{35}$$

We shall consider at first the behaviour of $T_{\alpha\beta\gamma\delta}$ as $S_{12} \to \infty$ and $S_{13} \approx -\mu^2$

and shall proceed from the assumption that

$$
\begin{aligned}
A(S_{12}; S_{13}, S_{23}) &\to S_{12}F_3(S_{13}),\\
A(S_{13}; S_{23}, S_{12}) &\to S_{12}F_2(S_{13}),\\
A(S_{23}; S_{12}, S_{13}) &\to S_{12}F_1(S_{13}).
\end{aligned}
\tag{36}
$$

It is well known that the Feynman amplitude $A(S_{12}; S_{13}, S_{23})$ is not an analytic function of S_{12} in the upper half plane. (The functions corresponding to the retarded commutators are analytic in the upper half plane.)Therefore if we pass through the upper half plane from S_{12} to $-S_{12}$ we shall obtain not $A(-S_{12}; S_{13}, 4\mu^2+S_{12}-S_{13})$ but $A^*(-S_{12}; S_{13}, 4\mu^2+S_{12}-S_{13})$. However $A^*(-S_{12}; S_{13}, 4\mu^2+S_{12}-S_{13}) \to S_{12}F_1^*(S_{13})$ as $S_{12} \to \infty$. Hence, in view of the fact that $S_{12}F_3(S_{13})$ as $S_{12} \to -S_{12}$ transforms into $-S_{12}F_3(S_{13})$ we obtain

$$F_1^*(S_{13}) = -F_3(S_{13}), \tag{37}$$

and in a similar way we shall obtain for $A(S_{11}; S_{12}, S_{23})$

$$F_2(S_{13}) = -F_2^*(S_{13}). \tag{38}$$

If both functions $F_1(S_{13})$ and $F_2(S_{13})$ were non-zero it would follow from (34) firstly that the backward scattering is of the same order as the forward for any α, β and secondly that the cross-section of the forward charge exchange is of the same order as elastic scattering. The second is evident; to make the first evident it is sufficient to consider $T_{\alpha\beta,\gamma\delta}$ as $S_{12} \to \infty$ and $S_{23} \approx -\mu^2$ and to put $\alpha = \gamma$, $\beta = \delta$. As $S_{12} \to \infty$ and $S_{23} \approx -\mu^2$ we shall have from (34)

$$T_{\alpha\beta,\alpha\beta} = S_{12}\{F_1(S_{23})+\delta_{\alpha\beta}F_3(S_{23})+\delta_{\alpha\beta}F_2(S_{23})\}. \tag{39}$$

It follows from (39) that in order to prevent particles of different charges from backward scattering with the same amplitude as for forward scattering it is necessary that $F_1(S_{23}) = 0$. Then it will follow from (37) that $F_3(S_{13}) = 0$. In this case we obtain as $S_{12} \to \infty$, $S_{13} \to -\mu^2$

$$T_{\alpha\beta,\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}S_{12}F_2(S_{13}), \tag{40}$$

i.e. only scattering without charge-exchange occurs in the forward direction. Thus if we assume that backward scattering of particles with different charges is comparatively small we come to the conclusion that pion-pion scattering is characterized at high energies by a single scaler function $A(S_{13}; S_{23}, S_{12})$ symmetrical with respect to S_{12} and S_{23} in exactly the same way as for neutral particles without spins.

The unitarity condition for the channel in which S_{13} is the energy (area III in fig. 1) is written in a way analogous to (12) for each of the three amplitudes corresponding to definite isobaric spins 0, 1 and 2.

$$\begin{aligned} T_0 &= 3A(S_{13}; S_{23}, S_{12}) + A(S_{12}; S_{13}, S_{23}) + A(S_{23}; S_{12}; S_{13}), \\ T_1 &= A(S_{12}; S_{13}, S_{23}) - A(S_{23}; S_{12}, S_{13}), \\ T_2 &= A(S_{12}; S_{13}, S_{23}) + A(S_{23}; S_{12}, S_{13}). \end{aligned} \tag{41}$$

Each of these relations can be continued into region $|z| > 1$. However, for $z \gg 1$, since $A(S_{12}; S_{23}, S_{13})$ and $A(S_{23}; S_{12}, S_{13})$ make a small contribution to scattering at high energy, it is sufficient to consider only the relation for T_0 which is identical with (13) and will lead to the same results as for neutral particles.

Another interesting consequence of these results is noteworthy. Under the same assumption as in section 3 we find that

$$A_1(S_{13}; S_{12}, S_{23}) = SB(\xi)f(t), \tag{42}$$

with $S = S_{12}$ and $t = S_{13}$.

If there were no $B(\xi)$ it would follow from (38) that

$$A = iSf(t), \tag{43}$$

i.e., the scattering amplitude is imaginary (the real part could be of the order of $1/S$ as compared with the imaginary).

In the presence of $B(\xi)$ the situation changes drastically. With the substitution of S for $-S$, $B(\xi)$ will be replaced by $B(\xi+i\pi)$. It is easy to show that in this case the correct expression for the whole amplitude instead of (43) will have the form

$$A(S, t) = iSB[\ln(-iS)]f(t). \tag{44}$$

The function $A(S, t)$ in this case cannot be imaginary. If $B(\xi) \approx q/\xi$ as $S \to \infty$, then

$$B[\ln(-iS)] \sim \frac{1}{\xi^q}\left[1+q\,\frac{i\pi}{2\xi}\right]. \tag{45}$$

Thus, the real part of $A(S, t)$ is small only logarithmically as compared with the imaginary part.

The main results obtained in the work can be summarized in the following way:

1) It is shown that the representation of the scattering amplitude at high energies in the form $Sf(t)$ which holds if the scattering is of diffractional character contradicts the unitarity condition.

2) It follows from the conditions of unitarity and analyticity under natural assumptions that the total cross-section for the scattering decreases at high energies.

The treatment does not claim to be mathematically rigorous, but in our opinion it is convincing enough from the physical standpoint to encourage further investigation.

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